On Probability of Commutative Pairs in a Group

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Abstract

Is there any tricky way to determine an abelian group? If we randomly choose a pair of elements in a non-abelian group, is there any upper bound for the probability that they commute? The answer is yes, and there's a sharp bound for it.

Lemma 1. *If $^G/_Z$ is cyclic, then G is abelian for any group G and its center Z.*

Proof. Let $^G/_Z \cong C_p$ with generator r. Then every element in G can be written as $r^i z$ for $i \in \mathbb{Z}_p$, $z \in \mathbb{Z}$.
Since $(r^i z_1)(r^j z_2) = r^i z_1 r^j z_2 = r^i r^j z_1 z_2 r^j r^i z_2 z_1 = r^j z_2 r^i z_1 = (r^j z_2)(r^i z_1)$
So G is abelian. ☐

Corollary 1. *If h is non-abelian, then $|\mathbb{Z}| \leq {}^{|\mathbb{G}|}/_4$.*

Proof. By lemma 1, $^{|G|}/_{|Z|} \neq 2$ or 3, otherwise $^G/_Z$ is cyclic so that G is abelian. Also notice that the equality can be achieved, for instance, D_8. ☐

Note that there is a direct approach to prove this corollary which does not require lemma 1 at all.

Proof. Consider an element g that is not in the center. We know that $Z < C_g < G$, and both of them are strict. ☐

Now we have enough tools to succeed.

Theorem 1. *Let G be a finite group of order n, and let p be the probability that a randomly chosen ordered pair of elements in G commutes. If $p > \frac{5}{8}$, then the group must be abelian. The bound is sharp.*

Proof. Consider the conjugation of G acting on itself. We know that the orbits partition G.
For a non-abelian group G, there are some elements that have conjugate class size larger than 1. Only those elements that are in the center have conjugate class of size 1.
For an element x has conjugate class of size $k > 1$, there are $|G| - \frac{|G|}{ccl(x)}$ elements that do not commute with it by orbit-stabilizer theorem.
For all elements that are in the same conjugate class of size $k > 1$, there are $k(|G| - \frac{|G|}{ccl(x))}$ non-commutative pairs. The probability that they contribute is $\frac{k-1}{|G|}$.
The probability of non-commutative pairs is in total $\frac{\sum_k (k-1)}{|G|}$, summing through all conjugate class sizes.
Notice that $\sum k = |G|$
To minimize fraction, we need:

1. the proportion of elements that are not in the center must be minimized
2. each conjugate class has size two so that their contribution to the non-commutative pairs $\frac{k-1}{k}$ is minimized

For 1, by lemma 1, we know that the proportion is at most $\frac{3}{4}$. So $\frac{\sum_k (k-1)}{|G|} \geq \frac{3}{8}$, so that there are at most $\frac{5}{8}$ ordered pairs that are commutative. For 2, it can be achieved by considering D_8, which is a sharp construction.

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